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AN EXPLICIT SOLUTION FOR THE LAGRANGE MULTIPLIERS ON THE SINGULAR SUBARC OF AN OPTIMAL TRAJECTORY

by Rowland E. Burns

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LIST OF SYMBOLS (Cont'd)

Latin Symbols

x, y, z	Cartesian coordinates
y	A general variable

Greek Symbols

γ	Direction angle of rocket thrust
Δ	A determinant
δ	Direction angle of rocket thrust
δ_{ij}	Kronecker delta ($\delta_{ij} = 0$ if $i \neq j$: $\delta_{ij} = 1$ if $i = j$)
λ	If subscripted, one of three Lagrange multipliers: if not subscripted, the magnitude of the vector Lagrange multiplier $\vec{\lambda}$
μ	Gravitational parameter of the attracting planet
ν	Lagrange multiplier
σ	Lagrange multiplier

Superscripts

$\cdot$	Denotes derivative with respect to time
T	Indicates the transpose of a vector or matrix
$\sim$	Indicates a matrix
$\rightarrow$	Indicates a vector
$*$	Indicates a modified form of a quantity

LIST OF SYMBOLS (Concluded)

Subscripts

o	Indicates an initial point ($t = t_o$)
f	Indicates a final point ($t = t_f$)
max	Indicates the maximum value of a quantity
min	Indicates the minimum value of a quantity
i, j	Summation indices
x, y, z	Indicates a cartesian component of some quantity

AN EXPLICIT SOLUTION FOR THE LAGRANGE MULTIPLIERS ON THE SINGULAR SUBARC OF AN OPTIMAL TRAJECTORY

SUMMARY

The problem of optimal thrust programming of a single stage rocket-powered vehicle to achieve maximum payoff at specified end conditions is discussed. The vehicle is assumed to operate in vacuo in an inverse square gravitational field. The Lagrange multipliers are analytically determined for the so-called singular arc and the mass flow rate to maintain a singular arc condition is derived. These Lagrange multipliers are directly related to the steering program by algebraic equations.

INTRODUCTION

An early problem posed by rocket technology to the mathematical disciplines was the determination of the rocket trajectory which maximized the difference between the initial and final values of some function of the trajectory end points. This problem, first treated by the calculus of variations, has been the subject of numerous research papers.

A large amount of work was done in attempts to optimize the flight profile as a function of time under the assumption that the magnitude of the thrust was constant. That is, the optimization was concerned only with determining the direction of thrust as a function of time, but not the magnitude of thrust as a function of time.

As the theoretical developments became more sophisticated, the case of variable thrust magnitude was also treated. In this case, the thrust is assumed to be bounded between upper and lower limits; the lower limit may be zero (a coasting arc) but the upper limit is usually assumed to be finite so that the possibility of impulsive velocity change is excluded.

Concurrently with these efforts, work progressed in directions which may be regarded as perturbational effects of the basic problem. Such effects include aerodynamic drag, non-spherical gravitational fields, etc. These effects are not part of the present study, and in this report the vehicle will be assumed to operate in vacuo over a spherical earth.

It is well known that if the formulation of the optimal trajectory problem allows variable thrusting maneuvers to occur, three types of subarcs, which are categorized by thrust level, may occur in the overall flight profile. The first possibility is arcs which have constant (non-zero) thrust levels over a non-zero portion of the trajectory. The second possible type of arc is the zero thrust arc. This possibility can be easily justified on an intuitive basis by considering an orbit transfer maneuver with a high thrust vehicle between widely separated orbits.

The third type of arc is the so-called singular subarc of an optimal trajectory. This arc occurs when the portion of the Hamiltonian which yields the direction or magnitude of thrust as a function of time (via Euler's equations) vanishes identically. In this case, the thrust level is not immediately available from the Hamiltonian and further work must be done to determine the magnitude of thrust as a function of time.

As long as the problem is formulated in the calculus of variations framework, Lagrange multipliers or equivalent auxiliary variables must be introduced to determine the direction of thrust as a function of time. Much of the difficulty in the numerical calculation of optimal trajectories arises from the fact that the initial values of these Lagrange multipliers are not known and the resulting split boundary value problem requires repeated integrations of the equations which result from the theory. Any technique which solves analytically for the multipliers (hence, thrust direction) as a function of the state variables, time, and constants of integration would greatly reduce the labor and time involved in numerical isolations of optimal trajectories. The above classification of the three types of arcs which may occur along an optimal trajectory is particularly well suited to statements about which cases allow such an analytic solution of the Lagrange multipliers. The case of zero thrust is simplest, and yielded such a solution before the other types. This solution has been known for several years [1, 2].*

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*W. E. Miner appears to be the first to have obtained this determination. The results of his work are reported in Aeroballistics Internal Note 20-63, Transformation of the λ-Vector and Closed Form Solution of the λ-Vector on Coast Arcs, George C. Marshall Space Flight Center, Huntsville, Alabama, 1963.

The case of constant (non-zero) thrust has not, as yet, been completely solved in the sense that is referred to here. A solution for three of six Lagrange multipliers in terms of state variables and constants of integration has been achieved through the use of a vectorial integral.

It is the third case that is treated in this paper. It is shown that for the case of the singular arc of an optimal trajectory, a complete analytic solution for the steering program is possible. The solution is demonstrated in the form of three simultaneous algebraic equations. Subsidiary equations then yield the additional Lagrange multipliers in terms of the solutions to these equations. Questions concerning the conditions under which these arcs actually do occur as segments of the trajectory are not discussed. For such considerations, the reader is referred to References 3, 4 and 5.

The analytic solution presented for the steering program is applicable to numerical trajectory calculations. Using these algebraic equations, it is not necessary to integrate the differential equations which normally are used to calculate the Lagrange multipliers over that portion of the trajectory which fulfills the definition of a singular subarc.

FORMULATION OF THE PROBLEM

We wish to determine the extremum value of the difference between the initial and final values of some function of the end points, G. For example, the fuel expenditure is minimized if we choose G as the difference between the initial mass, m_o, and the final mass, m_f, i.e.,

$$G = m_o - m_f .$$

In order to specify the constraints to which the rocket vehicle is subjected, we write the rocket equation as

$$\dot{\vec{V}} = \frac{T}{m} \hat{u}_T - \vec{g} \qquad (1)$$

where

$\vec{V}$ = velocity

T = thrust

m = mass

$\hat{u}_T$ = unit vector in the thrust direction

$\vec{g}$ = acceleration of gravity

The thrust, T, is assumed to satisfy the inequality

$$T_{min} \leq T \leq T_{max} \quad .$$

The velocity, $\vec{V}$, is related to the position vector, $\vec{r}$, by

$$\dot{\vec{r}} = \vec{V} \quad . \tag{2}$$

Finally, the thrust, T, and mass flow rate, mc, are related by

$$T = -\dot{m}c \quad . \tag{3}$$

Using the standard techniques of variational calculus, we now introduce vector Lagrange multipliers $\vec{\lambda}$ and $\vec{\nu}$ and a scalar multiplier σ to form the Lagrange fundamental function, F, as

$$F = \vec{\lambda} \cdot (\dot{\vec{V}} - T/m\hat{u}_T + \vec{g}) + \vec{\nu} \cdot (\dot{\vec{r}} - \vec{V}) + \sigma(\dot{m} + T/c) \quad . \tag{4}$$

The problem we now wish to examine is to determine the extremal values of

$$G^* = G + \int_{t_o}^{t_f} F \, dt \quad .$$

To initiate this study we first rewrite F in the more tractable scalar notation by introducing

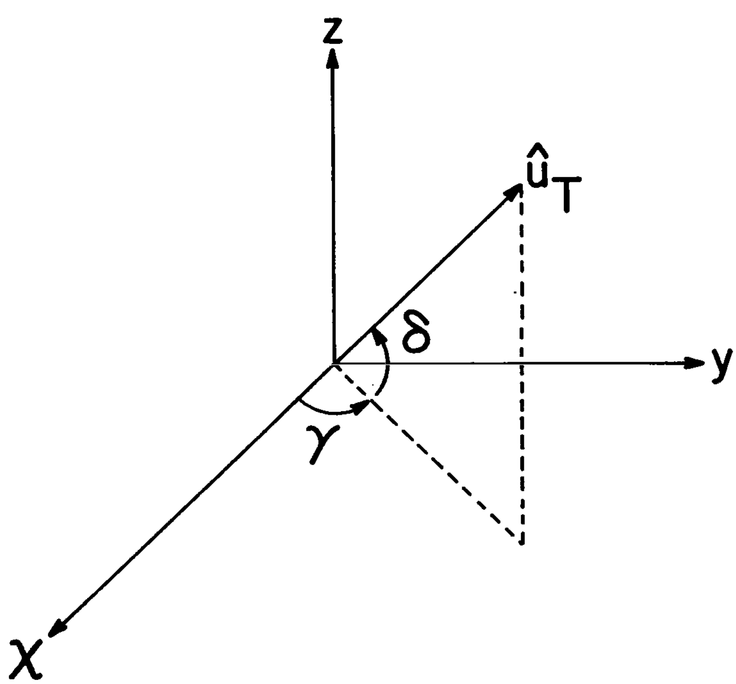
$$\vec{r} = (x, y, z) \tag{5}$$

$$\vec{V} = (V_x, V_y, V_z) \tag{6}$$

$$\hat{u}_T = (\cos\gamma \cos\delta, \ \sin\gamma \cos\delta, \ \sin\delta) \tag{7}$$

$$\vec{g} = (g_x, g_y, g_z) \tag{8}$$

where γ and δ are control angles (i. e. , direction angles of the vector $\hat{u}_T$) as shown in the figure below and subscripts indicate components, not partial derivatives.



We may now rewrite F as

$$F = \lambda_1(\dot{V}_x - T/m \cos\delta \cos\gamma + g_x) + \lambda_2(\dot{V}_y - T/m \cos\delta \sin\gamma + g_y)$$

$$+ \lambda_3(\dot{V}_z - T/m \sin\delta + g_z) + \nu_1(\dot{x} - V_x) + \nu_2(\dot{y} - V_y)$$

$$+ \nu_3(\dot{z} - V_z) + \sigma(T/c + \dot{m}) \quad . \tag{9}$$

The Euler-Lagrange equations, a necessary condition for an extremal path, are

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}_i}\right) - \frac{\partial F}{\partial y_i} = 0 \tag{10}$$

where the variables y_i are V_x, V_y, V_z, x, y, z, γ, δ, T, and m.

Applying the Euler-Lagrange equation to the variables V_x, V_y, and V_z we have

$$\dot{\lambda}_1 + \nu_1 = 0 \tag{11}$$

$$\dot{\lambda}_2 + \nu_2 = 0 \tag{12}$$

$$\dot{\lambda}_3 + \nu_3 = 0 \quad . \tag{13}$$

Defining the ordered set $(\lambda_1, \lambda_2, \lambda_3)$ as the vector $\vec{\lambda}^T$ and (ν_1, ν_2, ν_3) as the vector $\vec{\nu}^T$, we can write equations (11), (12), and (13) as

$$\dot{\vec{\lambda}} + \vec{\nu} = 0 \quad . \tag{14}$$

Turning our attention to the variables (x, y, z) for the Euler-Lagrange equations, we find

$$\dot{\nu}_1 - \lambda_1 \frac{\partial g_x}{\partial x} - \lambda_2 \frac{\partial g_y}{\partial x} - \lambda_3 \frac{\partial g_z}{\partial x} = 0$$

$$\dot{\nu}_2 - \lambda_1 \frac{\partial g_x}{\partial y} - \lambda_2 \frac{\partial g_y}{\partial y} - \lambda_3 \frac{\partial g_z}{\partial y} = 0$$

$$\dot{\nu}_3 - \lambda_1 \frac{\partial g_x}{\partial z} - \lambda_2 \frac{\partial g_y}{\partial z} - \lambda_3 \frac{\partial g_z}{\partial z} = 0 \quad .$$

These may be conveniently rewritten in matrix notation as

$$\begin{pmatrix} \dot{\nu}_1 \\ \dot{\nu}_2 \\ \dot{\nu}_3 \end{pmatrix} - \begin{pmatrix} \frac{\partial g_x}{\partial x} & \frac{\partial g_y}{\partial x} & \frac{\partial g_z}{\partial x} \\ \frac{\partial g_x}{\partial y} & \frac{\partial g_y}{\partial y} & \frac{\partial g_z}{\partial y} \\ \frac{\partial g_x}{\partial z} & \frac{\partial g_y}{\partial z} & \frac{\partial g_z}{\partial z} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = 0 \quad .$$

Using our previous notation for the vectors $\vec{\lambda}$ and $\vec{\nu}$, and abbreviating the matrix by $\underset{\sim}{A}^*$, we have

$$\dot{\vec{\nu}} - \widetilde{A}^* \vec{\lambda} = 0 \quad . \tag{15}$$

The Euler-Lagrange equation for variable γ yields

$$T/m(\lambda_1 \cos\delta \, \sin\gamma - \lambda_2 \cos\delta \, \cos\gamma) = 0 \quad .$$

For a powered arc, T/m is not equal to zero. If we assume that

$$\delta \neq \frac{(2n + 1)}{2} \pi$$

we obtain

$$\tan \gamma = \lambda_2/\lambda_1 \quad . \tag{16}$$

Similarly, for variable δ

$$T/m(\lambda_1 \sin \delta \cos \gamma + \lambda_2 \sin \delta \sin \gamma - \lambda_3 \cos \delta) = 0 \quad .$$

Again, for $T/m \neq 0$, we have

$$\tan \delta = \frac{\lambda_3}{\lambda_1 \cos \gamma + \lambda_2 \sin \gamma} \quad .$$

Eliminating γ from this equation by use of equation (16) we obtain

$$\tan \delta = \frac{\pm \lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2}} \quad . \tag{17}$$

It may now be seen that the variables γ and δ have been related to the Lagrange multipliers via equations (16) and (17).

The Euler-Lagrange equation cannot be applied directly to variable T, since T enters linearly. Writing the equation for F as

$$F = \lambda_1(\dot{V}_x + g_x) + \lambda_2(\dot{V}_y + g_y) + \lambda_3(\dot{V}_z + g_z) + \nu_1(\dot{x} - V_x)$$

$$+ \nu_2(\dot{y} + V_y) + \nu_3(\dot{z} - V_z) + \sigma\dot{m} + T(\sigma/c \mp \lambda/m) = 0$$

(where λ has been written for $\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}$) we see that F can become independent of T for certain situations; namely, if the coefficient of T vanishes

$$\sigma/c \mp \lambda/m = 0 \quad .$$

If the coefficient of T vanishes, the value of T becomes indeterminate and we may write (at the moment) only that T is bounded within the physical limits of the problem, i. e.,

$$T_{min} \leq T \leq T_{max} \quad .$$

The sign ambiguity in the expression $\sigma/c \mp \lambda/m = 0$ can be resolved by the Weierstrass test. Originally

$$\tan \gamma = \lambda_2/\lambda_1 \tag{16}$$

so that

$$\cos \gamma = \frac{\pm \lambda_1}{\sqrt{\lambda_1^2 + \lambda_2^2}}$$

and

$$\sin \gamma = \pm \frac{\lambda_2}{\sqrt{\lambda_1^2 + \lambda_2^2}}$$

with

$$\text{sign } (\cos \gamma) = \text{sign } (\sin \gamma) \quad .$$

Then

$$\tan \delta = \frac{\pm \lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2}}$$

giving

$$\text{sign}(\tan\delta) = \text{sign}(\cos\gamma) \quad .$$

Then

$$\cos\delta = \pm\sqrt{\frac{\lambda_1^2 + \lambda_2^2}{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}$$

and

$$\sin\delta = \pm\frac{\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}$$

with

$$\text{sign}(\cos\delta) = \text{sign}(\sin\delta) \quad .$$

In order to pursue the question of sign choice further, we may write F as

$$F = \lambda_1(\dot{V}_x + g_x) + \lambda_2(\dot{V}_y + g_y) + \lambda_3(\dot{V}_z + g_z) + \nu_1(\dot{x} - V_x)$$

$$+ \nu_2(\dot{y} - V_y) + \nu_3(\dot{z} - V_z) + \sigma\dot{m}$$

$$+ T[\sigma/m - 1/m(\lambda_1 \cos\delta\cos\gamma + \lambda_2\cos\delta\sin\gamma + \lambda_3\sin\delta)] \quad .$$

The Weierstrass condition now requires that for a minimum the function

$$E = (t, y, \dot{y}, y^*, \dot{y}^*) = F(t, y^*, \dot{y}^*) - F(t, y, \dot{y})$$

$$- \sum_i (\dot{y}^* - \dot{y}_i)\frac{\partial F}{\partial \dot{y}_i} \geq 0$$

be positive for all sets y^* sufficiently near y and for all sets $\dot{y}^*$. The last member of this equation vanishes under consideration of the strong variations of the controls γ and δ since $\dot{\gamma}$ and $\dot{\delta}$ do not appear. Thus

$$E = F(y^*, \dot{y}^*) - F(y, \dot{y})$$

$$= \sum_i \lambda_1 \left[\frac{\partial F(y^*, \dot{y}^*)}{\partial \lambda_i}\right] - \sum_i \lambda_1 \left[\frac{\partial F(y, \dot{y})}{\partial \lambda_i}\right] \quad .$$

For $y_i \neq \gamma$ and $y_i \neq \delta$ we have

$$y_i = y_i^*$$

so that

$$T/m(\lambda_1 \cos \delta \cos \gamma + \lambda_2 \cos \delta \sin \gamma + \lambda_3 \sin \delta)$$

$$\geq T/m(\lambda_1 \cos \delta^* \cos \gamma^* + \lambda_2 \cos \delta^* \sin \gamma^* + \lambda_3 \sin \delta^*) \quad .$$

Choosing

$$\delta = \delta^*, \quad \gamma = \gamma^* \quad \text{or} \quad \gamma = \gamma^* + \pi$$

we have

$$\cos \delta (\lambda_1 \cos \gamma + \lambda_2 \sin \gamma) \geq 0 \quad .$$

Then

$$\cos \delta \left(\pm \sqrt{\lambda_1^2 + \lambda_2^2}\right) \geq 0$$

or

$$\text{sign}\sqrt{\lambda_1^2 + \lambda_2^2} = \text{sign}\cos\delta \quad .$$

Choosing

$$\gamma^* = \gamma \;, \quad \delta^* = \delta \quad \text{or} \quad \delta^* = \delta + \pi$$

gives

$$\lambda_1 \cos\delta \cos\gamma + \lambda_2 \cos\delta \sin\gamma + \lambda_3 \sin\delta \geq 0$$

or

$$\pm\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} \geq 0$$

then

$$\text{sign}\left(\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}\right) = + \quad .$$

Our function F now becomes

$$F = \lambda_1(\dot{V}_x + g_x) + \lambda_2(\dot{V}_y + g_y) + \lambda_3(\dot{V}_z + g_z)$$

$$+ \nu_1(\dot{x} - V_x) + \nu_2(\dot{y} - V_y) + \nu_3(\dot{z} - V_z) + \sigma\dot{m} + T(\sigma/c - \lambda/m) = 0$$

and the condition for the singular arc is

$$\sigma/c - \lambda/m = 0 \quad . \tag{18}$$

The remaining sign ambiguity, $\pm\sqrt{\lambda_1{}^2+\lambda_2{}^2}$, corresponds to the physical choice of initial firing direction. This cannot be expected to be a result of mathematics. Further discussions will be restricted to the case of the singular arc.

Our final variable, m, gives the equation

$$\dot{\sigma} - T/m^2(\lambda_1 \cos\delta \cos\gamma + \lambda_2 \cos\delta \sin\gamma + \lambda_3 \sin\delta) = 0 \quad .$$

Again, expressing γ and δ in terms of λ_1, λ_2, and λ_3 we have

$$\dot{\sigma} \mp T/m^2\, \lambda = 0$$

which becomes - by our discussion of sign choice

$$\dot{\sigma} - T/m^2\, \lambda = 0 \quad . \tag{19}$$

Before proceeding to the solution for the multipliers, we note that from equation (14)

$$\dot{\vec{\lambda}} = - \vec{\nu}$$

so that equation (15) may be written

$$\ddot{\vec{\lambda}} = \widetilde{A}^*\vec{\lambda} = 0 \quad . \tag{20}$$

The matrix $\widetilde{A}^*$ is too general for our purposes. We specify a Keplerian field

$$\vec{g} = - \mu\, \frac{\vec{r}}{r^3}$$

where μ is the gravitational parameter MG, i.e., the mass of the attracting planet multiplied by the universal gravitational constant. From this definition

$$\frac{\partial g_j}{\partial x_i} = \frac{\partial}{\partial x_i}\left[\mu \frac{x_j}{(x^2 + y^2 + z^2)^{3/2}}\right]$$

$$= \mu \frac{\delta_{ij}}{(x^2 + y^2 + z^2)^{3/2}} - \mu \frac{3x_j x_i}{(x^2 + y^2 + z^2)^{5/2}}$$

$$= \mu \frac{\delta_{ij}}{r^3} - \mu \frac{3x_i x_j}{r^5} \quad (i, j = 1, 2, 3)$$

where

x_i = x or y or z

r has been written for $\sqrt{x^2 + y^2 + z^2}$

δ_{ij} is the Kronecker delta.

We can now write the expression $\tilde{A}^* \vec{\lambda}$ under the Keplerian field assumption, $\tilde{A}\vec{\lambda}$, as

$$\tilde{A}\vec{\lambda} = \begin{pmatrix} \frac{\mu}{r^3} - \frac{3\mu x^2}{r^5}, & \frac{-3\mu xy}{r^5}, & \frac{-3\mu xz}{r^5} \\ \frac{-3\mu xy}{r^5}, & \frac{\mu}{r^3} - \frac{3\mu y^2}{r^5}, & \frac{-3\mu yz}{r^5} \\ \frac{-3\mu xz}{r^5}, & \frac{-3\mu yz}{r^5}, & \frac{\mu}{r^3} - \frac{3\mu z^2}{r^5} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}$$

$$= \mu \begin{pmatrix} \frac{\lambda_1}{r^3} - \frac{3x^2\lambda_1}{r^5} - \frac{3xy\lambda_2}{r^5} - \frac{3xz\lambda_3}{r^5} \\ \\ -\frac{3xy\lambda_1}{r^5} + \frac{\lambda_2}{r^3} - \frac{3y^2\lambda_2}{r^5} - \frac{3yz\lambda_3}{r^5} \\ \\ -\frac{3xz\lambda_1}{r^5} - \frac{3yz\lambda_2}{r^5} + \frac{\lambda_3}{r^3} - \frac{3z^2\lambda_3}{r^5} \end{pmatrix}$$

$$= \mu \begin{pmatrix} \frac{\lambda_1}{r^3} \\ \\ \frac{\lambda_2}{r^3} \\ \\ \frac{\lambda_3}{r^3} \end{pmatrix} - 3\mu \begin{pmatrix} \frac{x}{r^5} \; (x\lambda_1 + y\lambda_2 + z\lambda_3) \\ \\ \frac{y}{r^5} \; (x\lambda_1 + y\lambda_2 + z\lambda_3) \\ \\ \frac{z}{r^5} \; (x\lambda_1 + y\lambda_2 + z\lambda_3) \end{pmatrix}$$

$$= \mu \left(\frac{\vec{\lambda}}{r^3} - 3 \frac{\vec{\lambda} \cdot \vec{r}}{r^5} \vec{r} \right) .$$

Equation (20) may now be written as

$$\ddot{\vec{\lambda}} + \mu \left(\frac{\vec{\lambda}}{r^3} - 3 \frac{\vec{\lambda} \cdot \vec{r}}{r^5} \vec{r} \right) = 0 \quad .$$

We may also rewrite the basic equation of motion, equation (1), into a form not involving the angles γ and δ by use of equations (7), (16), and (17). Thus

$$\ddot{\vec{r}} - \frac{T}{m} \frac{\vec{\lambda}}{\lambda} + \mu \frac{\vec{r}}{r^3} = 0 \quad .$$

The principal results of this section may now be summarized as

$$\ddot{\vec{r}} - \frac{T}{m}\frac{\vec{\lambda}}{\lambda} + \mu\frac{\vec{r}}{r^3} = 0 \quad , \tag{21}$$

$$\ddot{\vec{\lambda}} + \mu\left(\frac{\vec{\lambda}}{r^3} - 3\,\frac{\vec{\lambda}\cdot\vec{r}}{r^5}\,\vec{r}\right) = 0 \quad , \tag{22}$$

$$\dot{\sigma} - \frac{T\lambda}{m^2} = 0 \quad , \tag{23}$$

$$\frac{\sigma}{c} - \frac{\lambda}{m} = 0 \quad . \tag{24}$$

Equation (24) specifies that we remain on a singular arc.

Before proceeding to the integration of this system of equations, it is worthwhile to consider the following. In equation (21) we have three position and three velocity components which must be initially specified. In equations (22) and (23) we must specify the initial values of the vectors $\vec{\lambda}$, $\dot{\vec{\lambda}}$, and of the scalars σ and T (or $\dot{\sigma}$). This gives 14 initial conditions. Since the equations in the Lagrange multipliers are homogeneous, one initial value is arbitrary. Furthermore, equation (24) gives a relationship between λ and σ which must be maintained, thus reducing the number of initial conditions to 12. For a solution which yields the Lagrange multipliers as an algebraic relationship, we may expect to obtain six independent constants of integration.

DERIVATION OF INTEGRALS

From equation (23) we have

$$m\dot{\sigma} - \frac{T\lambda}{m} = 0 \quad , \tag{25}$$

and from equation (3)

$$\dot{m}\sigma + \frac{T\sigma}{c} = 0 \quad . \tag{26}$$

Using equation (24), equation (26) becomes

$$\dot{m}\sigma + \frac{T\lambda}{m} = 0 \quad . \tag{27}$$

Adding equations (25) and (27) we obtain

$$\frac{d}{dt}(m\sigma) = 0$$

or

$$m\sigma = C_1 \tag{28}$$

where C_1 is an integration constant.

To obtain our second integral, we note that the Euler-Lagrange equations are explicitly independent of time. From the general theory of variational calculus we know that

$$F - \sum_{i=1}^{10} \dot{y}_i F_{\dot{y}_i} = C_2$$

where

F is given by equation (9)

y_i represents V_x, V_y, V_z, x, y, z, γ, δ, T, or m.

Furthermore, since F is identically equal to zero, the above equation becomes

$$-\sum_{i=1}^{10} \dot{y}_i F_{\dot{y}_i} = C_2 \quad .$$

By differentiating and summation we obtain

$$\lambda_1 \dot{V}_x + \lambda_2 \dot{V}_y + \lambda_3 \dot{V}_z + \nu_1 \dot{x} + \nu_2 \dot{y} + \nu_3 \dot{z} + \sigma \dot{m} = -C_2 \quad .$$

Substituting $\dot{V}_x = \ddot{x}$, $\dot{V}_y = \ddot{y}$, $\dot{V}_z = \ddot{z}$, $\nu_1 = -\dot{\lambda}_1$, $\nu_2 = -\dot{\lambda}_2$, and $\nu_3 = -\dot{\lambda}_3$, this becomes

$$\lambda_1 \ddot{x} + \lambda_2 \ddot{y} + \lambda_3 \ddot{z} - \dot{\lambda}_1 \dot{x} - \dot{\lambda}_2 \dot{y} - \dot{\lambda}_3 \dot{z} + \sigma \dot{m} = -C_2 \quad ,$$

which may be written as

$$\vec{\lambda} \cdot \ddot{\vec{r}} - \dot{\vec{\lambda}} \cdot \dot{\vec{r}} + \sigma \dot{m} = -C_2 \quad . \qquad (29)$$

Taking the dot product of $\vec{\lambda}$ with $\ddot{\vec{r}}$ as given by equation (21) and inserting the result into equation (29) we find

$$\dot{\vec{\lambda}} \cdot \dot{\vec{r}} + \mu \frac{\vec{\lambda} \cdot \vec{r}}{r^3} - \left(\frac{T}{m} \lambda + \sigma \dot{m}\right) = C_2 \quad .$$

The term in parentheses of this equation may be written

$$\begin{aligned} \frac{T}{m} \lambda + \sigma \dot{m} &= \left(\frac{T\lambda}{m^2}\right) m + \sigma \dot{m} \\ &= \dot{\sigma} m + \sigma \dot{m} \\ &= \frac{d}{dt} (m\sigma) = 0 \quad , \end{aligned}$$

where equation (23) has been used. Thus, our second integral is

$$\vec{\lambda} \cdot \dot{\vec{r}} + \mu \frac{\vec{\lambda} \cdot \vec{r}}{r^3} = C_2 \quad . \tag{30}$$

Following Pines [6], we begin derivation of the third integral by noting that

$$\frac{d}{dt}(\dot{\vec{\lambda}} \cdot \vec{r}) = \dot{\vec{\lambda}} \cdot \dot{\vec{r}} + \ddot{\vec{\lambda}} \cdot \vec{r}$$

$$= \dot{\vec{\lambda}} \cdot \dot{\vec{r}} + \left(- \mu \frac{\vec{\lambda} \cdot \vec{r}}{r^3} + 3\mu \frac{\vec{\lambda} \cdot \vec{r}}{r^3} \right)$$

$$= \dot{\vec{\lambda}} \cdot \dot{\vec{r}} + 2\mu \frac{\vec{\lambda} \cdot \vec{r}}{r^3} \quad .$$

Substituting $\vec{\lambda} \cdot \dot{\vec{r}}$ from equation (30) we have

$$\frac{d}{dt}(\dot{\vec{\lambda}} \cdot \vec{r}) = C_2 + \mu \frac{\vec{\lambda} \cdot \vec{r}}{r^3} \quad . \tag{31}$$

Taking the dot product of equation (21) with $\vec{\lambda}$, the dot product of equation (22) with $\vec{r}$ and subtracting, gives

$$\frac{d}{dt}(\dot{\vec{\lambda}} \cdot \vec{r} - \vec{\lambda} \cdot \dot{\vec{r}}) = -\frac{T}{m}\lambda + 3\mu \frac{\vec{\lambda} \cdot \vec{r}}{r^3} \quad ,$$

so that

$$\mu \frac{\vec{\lambda} \cdot \vec{r}}{r^3} = \frac{1}{3}\frac{d}{dt}(\dot{\vec{\lambda}} \cdot \vec{r} - \vec{\lambda} \cdot \dot{\vec{r}}) + \frac{T}{3m}\lambda \quad .$$

Inserting $\mu \frac{\vec{\lambda} \cdot \vec{r}}{r^3}$ from this expression into equation (31) gives

$$\frac{d}{dt}(\dot{\vec{\lambda}} \cdot \vec{r}) = C_2 + \frac{1}{3}\frac{d}{dt}(\dot{\vec{\lambda}} \cdot \vec{r} - \vec{\lambda} \cdot \dot{\vec{r}}) + \frac{T}{3m}\lambda ,$$

or

$$\frac{d}{dt}(2\dot{\vec{\lambda}} \cdot \vec{r} + \lambda \cdot \dot{\vec{r}}) = 3C_2 + \frac{T}{m}\lambda .$$

Now, using the defining equation for T in terms of $\dot{m}$, equation (3), this may be written as

$$\frac{d}{dt}(2\dot{\vec{\lambda}} \cdot \vec{r} + \vec{\lambda} \cdot \dot{\vec{r}}) = 3C_2 - c\lambda \frac{\dot{m}}{m} . \tag{32}$$

To perform the integration we must first note that equations (24) and (28) yield

$$\lambda = \frac{C_1}{c} \tag{33}$$

making λ a constant. Equation (32) now integrates to yield

$$2\dot{\vec{\lambda}} \cdot \vec{r} + \vec{\lambda} \cdot \dot{\vec{r}} = 3C_2 t + C_1 \ell n \frac{m_o}{m} + C_3 . \tag{34}$$

To obtain our final integral, we return to equations (21) and (22). From these

$$\ddot{\vec{\lambda}} \times \vec{r} + \mu \frac{\vec{\lambda} \times \vec{r}}{r^3} = 0$$

$$\vec{\lambda} \times \ddot{\vec{r}} + \mu \frac{\vec{\lambda} \times \vec{r}}{r^3} = 0 .$$

Subtracting gives an immediately integrable equation yielding

$$\vec{\dot{\lambda}} \times \vec{r} - \vec{\lambda} \times \vec{\dot{r}} = \vec{C_4}$$

where $\vec{C_4} = (C_4,\ C_5,\ C_6)$ is a vector integration constant.

To determine if the constants C_4, C_5, and C_6 are independent, we apply the Jacobi test. Thus

$$C_4 = \dot{\lambda}_2 z - \dot{\lambda}_3 y - \lambda_2 \dot{z} + \lambda_3 \dot{y}$$

$$C_5 = \dot{\lambda}_3 x - \dot{\lambda}_1 z - \lambda_3 \dot{x} + \lambda_1 \dot{z}$$

$$C_6 = \dot{\lambda}_1 y - \dot{\lambda}_2 x - \lambda_1 \dot{y} + \lambda_2 \dot{x} \quad .$$

Now

$$\frac{\partial C_4}{\partial x} = 0 \ , \quad \frac{\partial C_4}{\partial y} = -\dot{\lambda}_3 \ , \quad \frac{\partial C_4}{\partial z} = \dot{\lambda}_2 \ , \quad \frac{\partial C_4}{\partial \dot{x}} = 0 \ , \quad \frac{\partial C_4}{\partial \dot{y}} = \lambda_3 \ , \quad \frac{\partial C_4}{\partial \dot{z}} = -\lambda_2$$

$$\frac{\partial C_5}{\partial x} = \dot{\lambda}_3 \ , \quad \frac{\partial C_5}{\partial y} = 0 \ , \quad \frac{\partial C_5}{\partial z} = -\dot{\lambda}_1 \ , \quad \frac{\partial C_5}{\partial \dot{x}} = -\lambda_3 \ , \quad \frac{\partial C_5}{\partial \dot{y}} = 0 \ , \quad \frac{\partial C_5}{\partial \dot{z}} = \lambda_1$$

$$\frac{\partial C_6}{\partial x} = -\dot{\lambda}_2 \ , \quad \frac{\partial C_6}{\partial y} = \dot{\lambda}_1 \ , \quad \frac{\partial C_6}{\partial z} = 0 \ , \quad \frac{\partial C_6}{\partial \dot{x}} = \lambda_2 \ , \quad \frac{\partial C_6}{\partial \dot{y}} = -\lambda_1 \ , \quad \frac{\partial C_6}{\partial \dot{z}} = 0$$

we may write

$$J = \begin{vmatrix} 0\ , & -\dot{\lambda}_3\ , & \dot{\lambda}_2\ , & 0\ , & \lambda_3\ , & -\lambda_2 \\ \dot{\lambda}_3\ , & 0\ , & -\dot{\lambda}_1\ , & -\lambda_3\ , & 0\ , & \lambda_1 \\ -\dot{\lambda}_2\ , & \dot{\lambda}_1\ , & 0\ , & \lambda_2\ , & -\lambda_1\ , & 0 \end{vmatrix} \quad .$$

A non-vanishing sub-determinant is

$$\begin{vmatrix} -\dot{\lambda}_3 , & \dot{\lambda}_2 , & 0 \\ 0 , & -\dot{\lambda}_1 , & \lambda_3 \\ \dot{\lambda}_1 , & 0 , & \lambda_2 \end{vmatrix} = -\dot{\lambda}_1(\lambda_2\dot{\lambda}_3 - \dot{\lambda}_2\lambda_3) \neq 0$$

in general, so our integration constants C_4, C_5, and C_6 are independent.

The preceding statement cannot be taken to imply that we may solve for $\dot{\lambda}_1$, $\dot{\lambda}_2$, and $\dot{\lambda}_3$ in terms of $\vec{\lambda}$, $\vec{r}$, $\dot{\vec{r}}$, and $\vec{C}_4$. To see this we write equation (35) as

$$\begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix} \begin{pmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{\lambda}_3 \end{pmatrix} = \begin{pmatrix} 0 & \dot{z} & -\dot{y} \\ -\dot{z} & 0 & \dot{x} \\ \dot{y} & -\dot{x} & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} + \begin{pmatrix} C_4 \\ C_5 \\ C_6 \end{pmatrix} .$$

Now

$$\Delta = \begin{vmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{vmatrix} = zxy - zxy \equiv 0$$

so that we cannot invert the coefficient matrix of $(\dot{\lambda}_1, \dot{\lambda}_2, \dot{\lambda}_3)^T$. This situation occurs in similar fashion with the angular momentum integral of the three-body problem.

DETERMINATION OF THE LAGRANGE MULTIPLIERS

The purpose of this section is to demonstrate three independent equations which involve Lagrange multipliers λ_1, λ_2, and λ_3 algebraically. The first such equation, equation (33), has already been derived. The second will be derived from equation (35). From this relationship we obtain

$$\vec{r} \cdot (\dot{\vec{r}} \times \vec{\lambda}) = \vec{r} \cdot \vec{C}_4 \tag{36}$$

since $\vec{r} \cdot (\vec{\dot{\lambda}} \times \vec{r}) \equiv 0$.

To obtain our third relationship we first note that from equation (35) we obtain

$$\vec{r} \times (\vec{\dot{\lambda}} \times \vec{r}) = \vec{\dot{r}} \times (\vec{C}_4 + \vec{\lambda} \times \vec{\dot{r}}) \ ,$$

or

$$(\vec{\dot{r}} \cdot \vec{r})\vec{\dot{\lambda}} = (\vec{\dot{r}} \cdot \vec{\dot{\lambda}})\vec{r} + \vec{\dot{r}} \times (\vec{C}_4 + \vec{\lambda} \times \vec{\dot{r}}) \ .$$

Inserting $\vec{\dot{r}} \cdot \vec{\dot{\lambda}}$ from equation (30) into this expression gives

$$(\vec{\dot{r}} \cdot \vec{r})\vec{\dot{\lambda}} = \left(C_2 - \mu \frac{\vec{\lambda} \cdot \vec{r}}{r^3}\right) \vec{r} + \vec{\dot{r}} \times (\vec{C}_4 + \vec{\lambda} \times \vec{\dot{r}}) \ . \tag{37}$$

Now taking the cross product of $\vec{r}$ with equation (35) gives

$$(\vec{r} \cdot \vec{r})\vec{\dot{\lambda}} = (\vec{r} \cdot \vec{\dot{\lambda}})\vec{r} + \vec{r} \times (\vec{C}_4 + \vec{\lambda} \times \vec{\dot{r}}) \ .$$

Introducing $\vec{r} \cdot \vec{\dot{\lambda}}$ from equation (34) gives

$$(\vec{r} \cdot \vec{r})\vec{\dot{\lambda}} = \frac{1}{2} \left(C_1 \ln \frac{m_o}{m} + 3\, C_2 t + C_3 - \vec{\lambda} \cdot \vec{\dot{r}}\right) \vec{r}$$

$$+ \vec{r} \times (\vec{C}_4 + \vec{\lambda} \times \vec{\dot{r}}) \ . \tag{38}$$

Eliminating $\vec{\dot{\lambda}}$ between equation (37) and (38) under the assumption that $\vec{r} \cdot \vec{\dot{r}} \neq 0$ gives

$$\left[\frac{1}{2} \left(C_1 \ln \frac{m_o}{m} + 3\, C_2 t + C_3 - \vec{\lambda} \cdot \vec{\dot{r}}\right) \vec{r} + \vec{r} \times (\vec{C}_4 + \vec{\lambda} \times \vec{\dot{r}})\right] (\vec{\dot{r}} \cdot \vec{r})$$

$$= \left[\left(C_2 - \mu \frac{\vec{\lambda} \cdot \vec{r}}{r^3}\right) \vec{r} + \vec{\dot{r}} \times (\vec{C}_4 + \vec{\lambda} \times \vec{\dot{r}})\right] (\vec{r} \cdot \vec{r}) \ .$$

This equation can now be regrouped to give the form

$$\left[r^2 \left(C_2 - \mu \frac{\vec{\lambda} \cdot \vec{r}}{r^3} \right) - \frac{1}{2} (\vec{r} \cdot \dot{\vec{r}}) \left(C_1 \ell n \frac{m_o}{m} + 3\, C_2 t + C_3 - \vec{\lambda} \cdot \dot{\vec{r}} \right) \right] \vec{r}$$

$$+ \dot{\vec{r}} \times (\vec{C}_4 + \vec{\lambda} \times \dot{\vec{r}})(\vec{r} \cdot \vec{r}) - \vec{r} \times (\vec{C}_4 + \vec{\lambda} \times \dot{\vec{r}})(\vec{r} \cdot \dot{\vec{r}}) = 0 \ . \tag{39}$$

The last two members of this equation can be written as

$$\dot{\vec{r}} \times \vec{C}_4 (\vec{r} \cdot \vec{r}) - \vec{r} \times \vec{C}_4 (\dot{\vec{r}} \cdot r) + [\dot{\vec{r}} (\vec{r} \cdot \vec{r}) - \vec{r} (\vec{r} \cdot \dot{\vec{r}})] \times (\vec{\lambda} \times \dot{\vec{r}})$$

$$= - \vec{C}_4 \times [\vec{r} \times (\dot{\vec{r}} \times \vec{r})] + [\vec{r} \times (\dot{\vec{r}} \times \vec{r})] \times (\vec{\lambda} \times \dot{\vec{r}})$$

$$= [\vec{r} \times (\dot{\vec{r}} \times \vec{r})] \times (\vec{C}_4 + \vec{\lambda} \times \dot{\vec{r}}) = [\vec{r} \times (\dot{\vec{r}} \times \vec{r})] \times (\dot{\vec{\lambda}} \times \vec{r})$$

$$= - (\dot{\vec{\lambda}} \times \vec{r}) \cdot (\dot{\vec{r}} \times \vec{r}) \vec{r} + [(\dot{\vec{\lambda}} \times \vec{r}) \cdot \vec{r}] (\dot{\vec{r}} \times \vec{r})$$

$$= - (\dot{\vec{\lambda}} \times \vec{r}) \cdot (\dot{\vec{r}} \times \vec{r}) \vec{r}$$

where we have used

$$(\dot{\vec{\lambda}} \times \vec{r}) \cdot \vec{r} \equiv 0 \quad .$$

Using the final result, equation (39) may be written as

$$\left[r^2 \left(C_2 - \mu \frac{\vec{\lambda} \cdot \vec{r}}{r^3} \right) - \frac{1}{2} (\vec{r} \cdot \dot{\vec{r}}) \left(C_1 \ell n \frac{m_o}{m} + 3\, C_2 t + C_3 \right) \right.$$

$$\left. - \vec{\lambda} \cdot \dot{\vec{r}} - (\dot{\vec{\lambda}} \times \vec{r}) \cdot (\dot{\vec{r}} \times \vec{r}) \right] \vec{r} = 0 \quad .$$

Since $\vec{r} \neq 0$, the coefficient must vanish. Introducing $\dot{\vec{\lambda}} \times \vec{r}$ back into the coefficient yields

$$r^2 \left(C_2 - \mu \frac{\vec{\lambda} \cdot \vec{r}}{r^3}\right) - \frac{1}{2} (\vec{r} \cdot \dot{\vec{r}}) \left(C_1 \ell n \frac{m_o}{m} + 3\, C_2 t + C_3 - \vec{\lambda} \cdot \dot{\vec{r}}\right)$$

$$- (\vec{C}_4 + \vec{\lambda} \times \dot{\vec{r}}) \cdot (\dot{\vec{r}} \times \vec{r}) = 0 \quad . \tag{40}$$

Equations (33), (36), and (40) now define the Lagrange multipliers λ_1, λ_2, and λ_3 in terms of $\vec{r}$, $\dot{\vec{r}}$, m, and the integration constants. The multiplier σ comes from equation (28) and ν_1, ν_2, ν_3 can be obtained from equations (11), (12), and (13) once $\dot{\vec{\lambda}}$ is computed from equation (35).

DERIVATION OF THE MASS FLOW RATE EQUATION

The preceding information is not sufficient for a complete solution of the problem for we do not yet know either $\dot{m}$ or T as a function of the problem variables. The necessary information is available, though obscure.

From equation (24), which is the requirement that the singular arc be followed, we have

$$\frac{\dot{\sigma}}{c} - \frac{\dot{\lambda}}{m} + \frac{\lambda \dot{m}}{m^2} = 0 \quad .$$

Inserting the values for $\dot{\sigma}$ and $\dot{m}$ we find that

$$\dot{\lambda} = 0$$

or λ is a constant which we have ascertained from equation (33). Furthermore

$$\vec{\lambda} \cdot \vec{\lambda} = \left(\frac{C_1}{c}\right)^2 \quad ,$$

so that

$$\dot{\vec{\lambda}} \cdot \vec{\lambda} = 0 \quad ,$$

and

$$\ddot{\vec{\lambda}} \cdot \vec{\lambda} = - \dddot{\vec{\lambda}} \cdot \dot{\vec{\lambda}} \quad . \tag{41}$$

From equation (22) we obtain

$$\ddot{\vec{\lambda}} \cdot \vec{\lambda} = - \dot{\vec{\lambda}} \cdot \dot{\vec{\lambda}}$$

$$= - \mu \left[\frac{\lambda^2}{r^3} - \frac{3(\vec{\lambda} \cdot \vec{r}\,)^2}{r^5} \right]$$

$$= - \mu \left[\frac{C_1^2}{r^3 c^2} - \frac{3(\vec{\lambda} \cdot \vec{r}\,)^2}{r^5} \right] \quad . \tag{42}$$

The derivative of equation (42) is

$$2 \dot{\vec{\lambda}} \cdot \ddot{\vec{\lambda}} = + \mu \left[- 3 \left(\frac{C_1}{c} \right)^2 \frac{\vec{r} \cdot \dot{\vec{r}}}{r^5} - \frac{6 (\vec{\lambda} \cdot \vec{r}\,) (\vec{\lambda} \cdot \dot{\vec{r}} + \dot{\vec{\lambda}} \cdot \vec{r}\,)}{r^5} \right.$$

$$\left. + \frac{15 (\vec{\lambda} \cdot \vec{r}\,)^2 (\vec{r} \cdot \dot{\vec{r}}\,)}{r^7} \right] \quad , \tag{43}$$

but from equations (41) and (22)

$$\dot{\vec{\lambda}} \cdot \ddot{\vec{\lambda}} = 3 \mu \frac{\vec{\lambda} \cdot \vec{r}}{r^5} (\dot{\vec{\lambda}} \cdot \vec{r})$$

so that equation (43) may be written

$$r^2 \left[\left(\frac{C_1}{c} \right)^2 + 2(\vec{\lambda} \cdot \vec{r})(\vec{\lambda} \cdot \dot{\vec{r}} + 2\dot{\vec{\lambda}} \cdot \vec{r}) \right]$$

$$- 5(\vec{\lambda} \cdot \vec{r})^2 (\vec{r} \cdot \dot{\vec{r}}) = 0 \quad .$$

By use of equation (34), the previous equation may be written

$$r^2 \left[\left(\frac{C_1}{c} \right)^2 (\vec{r} \cdot \dot{\vec{r}}) + 2(\vec{\lambda} \cdot \vec{r}) \left(3C_2 t + C_1 \ell n \frac{m_o}{m} + C_3 \right) \right]$$

$$- 5(\vec{\lambda} \cdot \vec{r})^2 (\vec{r} \cdot \dot{\vec{r}}) = 0 \quad . \tag{44}$$

The derivative of equation (44) is

$$2(\vec{r} \cdot \dot{\vec{r}})^2 \lambda^2 + \left(C_1 \ell n \frac{m_o}{m} + 3C_2 t + C_3 \right) \Bigg\{ r^2 \left(\vec{\lambda} \cdot \dot{\vec{r}} + C_1 \ell n \frac{m_o}{m} + 3C_2 t + C_3 \right)$$

$$- (\vec{\lambda} \cdot \vec{r})(\vec{r} \cdot \dot{\vec{r}}) - 5(\vec{\lambda} \cdot \vec{r})(\vec{\lambda} \cdot \dot{\vec{r}})(\vec{r} \cdot \dot{\vec{r}}) + 6 C_2 r^2 (\vec{\lambda} \cdot \vec{r})$$

$$+ [r^2 \lambda^2 - 5(\vec{\lambda} \cdot \vec{r})^2] \left[\dot{\vec{r}} \cdot \dot{\vec{r}} - \frac{\mu}{r} \right] \Bigg\}$$

$$= \vec{\lambda} \cdot \vec{r} \left(\frac{\dot{m}}{m} \right) \left(\frac{c}{\lambda} \right) [3 \lambda^2 r^2 - 5(\vec{\lambda} \cdot \vec{r})^2] \quad . \tag{45}$$

Now $c \neq 0$ for a powered arc, and assuming

$$\vec{\lambda} \cdot \vec{r} \neq 0$$

we can solve for $\dot{m}$, with m given by equation (44), if

$$3\lambda^2 r^2 - 5(\vec{\lambda} \cdot \vec{r})^2 \neq 0 \quad . \qquad (46)$$

From equation (42), as noted by Corben [7], we may write

$$r^5 \dot{\vec{\lambda}} \cdot \dot{\vec{\lambda}} = \mu[\lambda^2 r^2 - 3(\vec{\lambda} \cdot \vec{r})^2] \quad .$$

Since the left side of this equation is non-negative, we can write the angle between $\vec{\lambda}$ and $\vec{r}$ as α and conclude

$$1 - 3 \cos^2 \alpha \geq 0$$

or

$$\frac{1}{3} \geq \cos^2 \alpha \ .$$

Our criterion of equation (46) becomes

$$\cos^2 \alpha \neq 3/5 \quad .$$

Since this condition is excluded by the previous inequality, we need only insure that

$$\vec{\lambda} \cdot \vec{r} \neq 0$$

to solve for $\dot{m}$; m is given by equation (44).

CONCLUSIONS

The general problem of an optimal rocket trajectory in vacuo in an inverse square gravitational field is composed of three possible types of segments. The first of these is the coasting arc, and, for this case, a complete determination of the Lagrange multipliers as algebraic functions of the state variables, time, and integration constants has been obtained [1, 2].

The second case, sub-arcs of constant thrust, is most important from the practical standpoint and has not, as yet, yielded an analytical solution.

The third case is sub-arcs of optimal intermediate thrust (singular sub-arcs) and this area has been treated in this report. It has been shown that integrals which exist in the literature, equations (28), (30), (34), and (35), are sufficient to explicitly yield the set of Lagrange multipliers which govern the guidance law along an optimal trajectory. The simultaneous algebraic equations governing the Lagrange multipliers associated with dynamical constraints are equations (33), (36), and (40). Once this set of equations has been solved, equations (11), (12), (13), and (28) then yield the additional multipliers which were associated with kinematical constraints in the original formulation. Equations (16) and (17) relate the Lagrange multipliers to the thrust direction angles. Finally, equation (45) supplies necessary information about the mass flow rate which is required to maintain the singular arc condition. Using these equations it is possible to compute an optimal singular subarc without encountering the difficulties associated with the usual split boundary value problem.

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